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LETTER TO THE EDITOR

Discussion of "First-order infinitesimal mechanisms", *Int. J. Solids Structures*, Vol. 27, pp. 505–515, (1991)

Compared to their earlier paper (Pellegrino and Calladine, 1986), this work is indeed an advance: it recognizes and employs the previously missing key concepts of the analysis, such as quadratic forms and stability. As a result, the previous effort is brought to fruition as a workable method for the detection of first-order infinitesimal mechanisms among pin-bar assemblies. More importantly, progress is made in treating assemblies with a degree of statical indeterminacy $s > 1$. Although the idea of constructing a linear combination of quadratic forms corresponding to independent states of self-stress has been suggested on several occasions in the prior literature, the proposed computational scheme is a welcome new development for structural mechanics.

Still, some points in the paper warrant a few comments.

(1) In Section 1, "the essence of Kuznetsov's criticism of the previous work" is vaguely interpreted to mean that "assemblies with two or more independent mechanisms... warrant a more formal procedure". More precisely, the object of the criticism was the "modified equilibrium matrix" which, as a rule, is of full rank, hence useless. (The exception—a globally statically indeterminate mechanism—is rather exotic and detectable by much simpler means.) As a result, construction of this matrix and determination of its rank had to be "supplemented" by some "sign check". The latter, when properly implemented in the subject paper, turned into the construction and investigation for sign definiteness of a certain quadratic form. This form happens to be the one known and used in the prior literature, and is alone sufficient for solving the problem at hand.

(2) In Section 3 the authors note that "the introduction of product forces enables us to avoid the complications of the standard second-order analysis". What complications? Obviously, a phenomenon rooted mathematically in a constrained strict minimum and physically in the concept of stability, necessitates a second-order analysis, and it was exactly its absence that doomed the "matrix analysis" in Pellegrino and Calladine (1986). When constructing the required *quadratic* form in the present paper, the authors are avoiding not a second-order analysis *per se*, but only the existing straightforward version of it. Indeed, their forming of equilibrium eqns (4) in Pellegrino and Calladine (1986) with nodal coordinates as variables, replaces the first differentiation of the constraint functions. Then the second differentiation appears as an incrementation of the nodal coordinates in the above equations when evaluating the product forces (24).

Serendipitously, it is the "standard second-order analysis", not the proposed method, that avoids the second differentiation; instead, it capitalizes on the fact that constraint functions for a pin-bar assembly are quadratic polynomials of the form $\Delta X^2+\Delta Y^2+\Delta Z^2-L^2 = 0$, with Δ denoting the Cartesian coordinate difference and L—the bar length. As a result, the quadratic form in all variables is obtained instantly, *without any calculation*, as a linear combination of constraint functions weighted by their respective tension coefficients. This is one of those cases where a more rigorous procedure is also the simplest to implement. The only remaining operation—elimination of dependent variables—is common to both approaches and, as the paper acknowledges, "exactly the same quadratic forms are obtained".

(3) The authors write in Section 5: "A comparison of Kötter's results with subsequent publications by other authors shows that little progress has been made over the past 75 years, in spite of several, intermittent attempts. In this paper we have shown that... the mechanisms are first-order infinitesimal if and only if there exists [a sign-definite linear

combination of quadratic forms]". Add another year with little progress: the above statement, central to the paper, does not hold for a system obtained by inserting a bar between pins 2 and 4 of Fig. 5. A sign-definite combination of quadratic forms *does not exist*, yet the system *is* an infinitesimal mechanism.

(4) One of the concluding remarks in the paper reads: "Our method has two obvious (*sic*!) advantages over methods proposed previously. First, our scheme makes use of physically-based quantities, e.g. mechanisms, states of self-stress, etc. rather than a second-order analysis of constraint equations in the manner of Kuznetsov.... Second, our scheme provides for assemblies with [$s \geq 1$], as in Levi-Civita and Amaldi (1930): However, we require much smaller matrices for our analysis".

Regarding the first of these statements, it is inexplicable that the authors decided to claim the advantage of "physically-based quantities" which were introduced much more systematically and with more depth and detail in the prior literature. For example, self-stress or, more precisely, its statical possibility, has always been a key concept in the "standard" analysis. Moreover, the crucial notion of stability of the state of self-stress (not stability "of all inextensional deformations") was highlighted, and linked to the positive definiteness of the quadratic or higher-order form expressing the lowest-order virtual work. Another key concept is that of perturbation (product) forces, i.e. forces capable of perturbing the system kinematically. The concept was introduced in Kuznetsov (1973) as the orthogonal complement of the column space of the equilibrium matrix, and employed in the statical analysis of underconstrained systems. If anything, the use of some of these conventional concepts by the authors is at times very confusing. Thus, in Pellegrino and Calladine (1986) the number of internal mechanisms, *im*, is defined as the number of independent displacements; in the present paper the authors imply that by "all mechanisms" they mean all *linear combinations* of independent mechanisms (hence, the appearance of the previously missing quadratic form).

As to the second of the stated advantages of the method, the proposed scheme involves two distinct features: a method for solving the generic problem with $s = 1$; and an algorithm dealing with linear combinations of quadratic forms for $s > 1$. While the authors deserve compliments for the progress made with the latter item, this development is not unique to their method. Furthermore, it is not clear what is meant by "much smaller matrices". If the authors claim credit for the elimination of dependent variables, the credit is not with them. Although Levi-Civita did not care to do that, all other researchers in the field, starting with Kötter, worked with independent displacements and smaller matrices. Ironically, this might be a reflection of the prevailing pre-computer mentality; today it would be more appropriate to use efficient computational means to immediately establish sign-definiteness of a quadratic form subject to linear constraints. This would make the evaluation of dependent variables unnecessary.

Incidentally, the fact that the scheme by Levi-Civita is formally free from statical consideration does not mean that "it poses no extra difficulty if $s > 1$"; the difficulty remains exactly the same and, as stated before, the authors should be praised for the progress achieved. The absence of a statical aspect in Levi-Civita's analysis is not an accident. The entire issue of kinematic mobility is purely geometric and, strictly speaking, is absolutely independent of statics. Thus, statical concepts related to kinematic analysis, such as self-stress and its stability, tension coefficients (force densities), perturbation (product) forces, virtual work and so on, are just interpretations of certain geometric facts in terms of statics. This is an expression of the statical–kinematic duality which, in turn, is a manifestation of the more general duality found in mathematical programming.

Turning from the two "obvious advantages" of the proposed method, two disadvantages can be noted as follows:

(i) By ignoring the ultimate source of information, the constraint equations (which for pin-bar assemblies are so invitingly simple), the proposed method is limited to detecting only first-order mechanisms, while dumping all higher-order ones into the same category as finite mechanisms.

(ii) Semi-formal techniques, such as writing analytical equilibrium equations and incrementing them "manually", are not conducive to using the available computer software with analytical differentiation capability.

CONCLUSION

The concept of perturbation forces proved instrumental in the *statical* analysis of underconstrained systems. Judging by its implementation and performance in the proposed method, the concept, apparently, is not as helpful in the *kinematic* analysis. The related concept of a modified equilibrium matrix has proved to be outright counterproductive in the kinematic analysis. As a result, both of the stated advantages of the proposed method over the "standard second-order analysis" are illusory, while the disadvantages are quite tangible. On the other hand, the outlined computational scheme dealing with a combination of quadratic forms is of interest, and its application to structural mechanics appears to be a new one.

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